16.1

Ehrenfest's theorem relates the time evolution of expectation values of Quantum operators to classical equations of motion. To determine how this theorem changes when the observable has explicit time dependence, I'll start with the time derivative of the expectation value of an operator $\hat{A}(t)$

The expectation value of $\hat{A}(t)$ is given by:

$$\langle \hat{A}(t) \rangle = \langle \psi(t) | \hat{A}(t) | \psi(t) \rangle$$

Taking the time derivative :

$$\frac{d}{dt} \langle \hat{A}(t) \rangle = \frac{d}{dt} \langle \psi(t) | \hat{A}(t) | \psi(t) \rangle$$

Using the product rule for differentiation :

$$\frac{d}{dt} \langle \hat{A}(t) \rangle = \langle \psi(t) | \hat{A}(t) | \psi(t) \rangle + \langle \psi(t) | \frac{d\hat{A}(t)}{dt} | \psi(t) \rangle + \langle \psi(t) | \hat{A}(t) | \psi(t) \rangle$$

From the Schrödinger equation, we know :

$$| \psi(t) \rangle = -\frac{i}{\hbar} \hat{H} | \psi(t) \rangle$$

$$\langle \psi(t) | = \frac{i}{\hbar} \langle \psi(t) | \hat{H}$$

Substituting these expressions :

$$\frac{d}{dt} \langle \hat{A}(t) \rangle = \frac{i}{\hbar} \langle \psi(t) | \hat{H} \hat{A}(t) | \psi(t) \rangle + \langle \psi(t) | \frac{d\hat{A}(t)}{dt} | \psi(t) \rangle - \frac{i}{\hbar} \langle \psi(t) | \hat{A}(t) \hat{H} | \psi(t) \rangle$$

Rearranging:

$$\frac{d}{dt}\langle \hat{A}(t) \rangle = \frac{i}{\hbar} \langle \psi(t) | [\hat{H}, \hat{A}(t)] | \psi(t) \rangle + \langle \psi(t) | \frac{d\hat{A}(t)}{dt} | \psi(t) \rangle$$

This can be written more compactly as:

$$\frac{d}{dt}\langle \hat{A}(t) \rangle = \frac{i}{\hbar} \langle [\hat{H}, \hat{A}(t)] \rangle + \langle \frac{d\hat{A}(t)}{dt} \rangle$$

16.2

The density matrix ρ for a quantum system is defined as:

$$\rho = \sum_n p_n |\psi_n\rangle\langle\psi_n|$$

where p_n are probabilities (non-negative and sum to 1) and $|\psi_n\rangle$ are pure states. To show that $Tr(\rho^2) \leq 1$, I'll start by expanding ρ^2:

$$\rho^2 = \left(\sum_n p_n |\psi_n\rangle\langle\psi_n|\right)\left(\sum_{n'} p_{n'} |\psi_{n'}\rangle\langle\psi_{n'}|\right)$$

$$\rho^2 = \sum_n \sum_{n'} p_n p_{n'} |\psi_n\rangle\langle\psi_n|\psi_{n'}\rangle\langle\psi_{n'}|$$

Now, taking the trace:

$$Tr(\rho^2) = \sum_n \sum_{n'} p_n p_{n'} Tr(|\psi_n\rangle\langle\psi_n|\psi_{n'}\rangle\langle\psi_{n'}|)$$

Using the property $Tr(|a\rangle\langle b|) = \langle b|a\rangle$:

$$Tr(\rho^2) = \sum_n \sum_{n'} p_n p_{n'} \langle\psi_{n'}|\psi_n\rangle\langle\psi_n|\psi_{n'}\rangle$$

$$Tr(\rho^2) = \sum_n \sum_{n'} p_n p_{n'} |\langle\psi_n|\psi_{n'}\rangle|^2$$

For orthonormal states, $|\langle\psi_n|\psi_{n'}\rangle|^2 = \delta_{nn'}$ which simplifies to:

$$Tr(\rho^2) = \sum_n p_n^2$$

Since p_n are probabilities, $0 \leq p_n \leq 1$ for all n, which means $p_n^2 \leq p_n$.

Therefore:

$$Tr(\rho^2) = \sum_n p_n^2 \leq \sum_n p_n = 1$$

This proves that $Tr(\rho^2) \leq 1$. Equality holds if and only if one of the p_n is 1 and all others are 0, which corresponds to a pure state.

$\boxed{16.7}$ The circuit shown in A1.21 consists of several quantum gates arranged in a specific pattern To determine what this circuit does, I'll analyze its matrix representation.

The circuit can be represented as a product of matrices, where each matrix corresponds to a quantum gate.
From the matrix multiplication shown in the problem:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$